Fermi National Accelerator Laboratory

FN-336
0402.000

On an Invariant of a System affected by Intra-Beam Scattering

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Fermilab, March , 81

1. Introduction

A. Piwinski has shown[1] that there is an invariant that is useful in understanding the effects of intra-beam scattering on betatron oscillations and the energy spread. After some mathematical manipulations, the invariant can be written as

$$\frac{1}{n}\left(\frac{1}{\gamma^2}-\alpha\right)\left\langle\left(\frac{\Delta p}{p}\right)^2\right\rangle + \frac{1}{\beta_x}\langle\epsilon_x\rangle+\frac{1}{\beta_y}\langle\epsilon_y\rangle = \text{const} \qquad (1 - 1)$$

where $p, \Delta p$ are the momentum and momentum deviation, $\pi\epsilon_x$ and $\pi\epsilon_y$ are the betatron emittances for horizontal and vertical directions, respectively, β_x and β_y are the horizontal and vertical betatron amplitude functions, γ is the particle energy in units of its rest energy, α is the momentum compaction factor ($=\overline{\eta^2/\beta_x^2}$ which is considered by Piwinski as a good approximation), η is the momentum dispersion function, $\langle A \rangle$ is the mean value of a quantity A over all particles, $\bar{A}$ is the the mean value of a quantity A in the orbit, and the integer n is 2 for an unbunched beam.

If we asume that betatron oscillations and energy oscillations are harmonic oscillators and that intra-beam scattering is a local elastic collision, we can derive the expression for this invariant more easily with the aid of the energy conservation law.

2. System of N three-dimensional harmonic oscillators

According to the above assumption, we can regard a beam as an isolated system of N three-dimensional harmonic oscillators which experience many random collisions among themselves. Here the longitudinal oscillation is taken as a free motion because we consider the case of an unbunched beam.

Consider the behavior of a single particle before and after a collision. Its behavior can be described by the Hamiltonian

$$H(x,P_x;Y,P_Y;\psi,\delta;\theta)=\frac{\nu_x}{2}(P_x^2+x^2)+\frac{\nu_Y}{2}(P_Y^2+Y^2)-\lambda x\delta+\frac{\nu_s}{2}\delta^2 \qquad (2-1)$$

where ν_x,ν_Y,ν_s are the horizontal and vertical betatron tunes ,and longitudinal frequency, $x,P_x;Y,P_Y;\psi,\delta$ are the canonical variables for horizontal, vertical ,and longitudinal motions ($\delta\equiv\Delta p/p$), λ is a coupling coefficient, and θ = s/R where R is the mean machine radius and s is the distance along the orbit.

We can separate the horizontal excursion of Eq. (2-1) into two parts, the equilibrium orbit and the homogeneous harmonic oscillation around the equilibrium orbit. This homogeneous harmonic oscillation corresponds to a pure betatron oscillation and the equilibrium orbit corresponds to the well-known closed orbit which varies linearly with δ , the longitudinal momentum deviation. The equation of motion obtained from the Hamiltonian (2-1) is then

$$\ddot{X} = -\nu_x^2 X + \lambda \nu_x \delta \qquad \left(\dot{} \equiv \frac{d}{d\theta} \right) \tag{2 - 2}$$

Therefore, the equilibrium orbit (X_{eq} ,P_{eq}) is written as

$$\begin{aligned} X_{eq} &= D\delta \qquad\qquad (D \equiv \lambda/\nu_x) \\ P_{eq} &= 0 \end{aligned} \tag{2 - 3}$$

Transforming into a new canonical variable (χ , P_χ),

$$\begin{aligned} \chi &= X - X_{eq} \\ P_\chi &= P_x - P_{eq} \end{aligned}$$

we find, from the generating function,

$$g(\chi , P_x ; \theta) = -(X_{eq} + \chi) P_x + P_{eq} \cdot \chi \tag{2 - 4}$$

the new Hamiltonian

$$\begin{aligned} K &= H + \frac{\partial g}{\partial \theta} \\ &= \frac{\nu_x}{2}[P_\chi^2 + (\chi + D\delta)^2] - \lambda(\chi + D\delta)\delta + \frac{\nu_s}{2}\delta^2 + \frac{\nu_Y}{2}[P_Y^2 + Y^2] \\ &= \frac{\nu_x}{2}[P_\chi^2 + \chi^2] + \frac{\nu_Y}{2}[P_Y^2 + Y^2] + \frac{1}{2}(\nu_s - \nu_x D^2)\delta^2 \end{aligned} \tag{2 - 5}$$

In terms of action-angle variables, we can write the Hamiltonian in the form

$$G = \nu_x J_\chi + \nu_Y J_Y + \frac{1}{2}(\nu_s - \nu_x D^2) J_s \tag{2 - 6}$$

Next, consider the behavior of the system before and after a collision. This behavior is described by the Hamiltonian H_{total}

$$H_{total} = \sum_{i=1}^{N} [\nu_x J_\chi + \nu_Y J_Y + \frac{1}{2}(\nu_s - \nu_x D^2) J_s]_i \tag{2 - 7}$$

Since we have assumed that the Coulomb interaction between paticles is a local elastic collision, the Hamiltonian H_{total} must be an invariant of the motion. We therefore obtain easily the invariant expression

$$\nu_x \langle J_x \rangle + \nu_Y \langle J_Y \rangle + \frac{1}{2}(\nu_s - \nu_x D^2)\langle J_s \rangle = \mathrm{const} \qquad (2-8)$$

where $\langle A \rangle$ is the mean value of A over all particles.

Finally, we can rewrite this expression in terms of betatron parameters. Relations between the parameters used above and the orbit parameters of a real machine are [2]

$D = \eta(s)/\sqrt{\beta(s)}$ because $x(s) = \sqrt{\beta_x(s)}\,X \longrightarrow x_{eq}(s) = \sqrt{\beta_x(s)}\,X_{eq} \longrightarrow \eta(s) = \sqrt{\beta_x(s)}\,D$

where $x(s)$, $x_{eq}(s)$ are the horizontal excursion and the equilibrium orbit in a real ring,

$\eta(s)$ is the momentum dispersion function.

$\nu_{x,Y} = R/\overline{\beta}_{x,y}$ (smooth approximation)

$\nu_s = R/\gamma^2$ because in the rest frame (independent variable: s) the momentum deviation takes the term $\Delta p/\gamma$ (See Appendix) and in the θ frame (independent variable: θ) the longitudinal energy must be multiplied by R.

Thus Eq. (2-8) becomes

$$R\frac{\langle J_x \rangle}{\overline{\beta}_x} + R\frac{\langle J_y \rangle}{\overline{\beta}_y} + \frac{R}{2}\left(\frac{1}{\gamma^2} - \frac{\eta^2(s)}{\overline{\beta}_x\,\beta_x(s)}\right)\langle J_s \rangle = \mathrm{const} \qquad (2-9)$$

Using the approximate expression for the momentum compaction factor [2]

$$\alpha = \frac{\eta^2(s)}{\overline{\beta}_x\,\beta_x(s)}$$

we obtain, from Eq.(2-9),

$$\frac{\langle J_x \rangle}{\bar{\beta}_x} + \frac{\langle J_y \rangle}{\bar{\beta}_y} + \frac{1}{2}\left(\frac{1}{\gamma^2} - \alpha\right)\langle J_s \rangle = \text{const} \qquad (2 - 10)$$

which is identical with the original form, Eq.(1-1), derived by Piwinski.

3. Conclusion

Although particles in the beam undergo many collisions, the invariant Eq.(2-10) is valid if the interaction is elastic and this is simply a consequence of the conservation of energy.

Aknowledgements

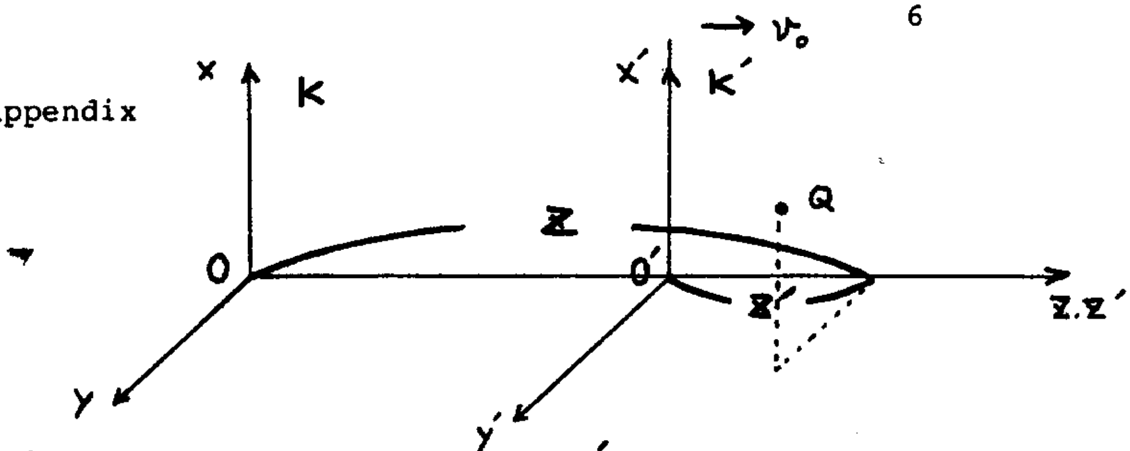
The auther is grateful to S. Ohnuma, F. Cole and A. Ruggiero for valuable suggestions and comments.

References:

(1) A. Piwinski, " INTRA-BEAM SCATTERING ", Proc. 9th Int. Conf. on High Energy Accelerators, p. 405 (1974)

(2) M. Sands, " THE PHYSICS OF ELECTRON STORAGE RINGS—AN INTRODUCTION", SLAC-121, UC-283(ACC) , p.77 (1970)

Appendix



K is the laboratory system and K′ is the system at rest with the synchronous particle which has the velocity $v_0\,(=c\beta_0)$ in K. The space-time coordinates of a test particle are (x,y,z,ict) in K and (x′,y′,z′,ict′) in K′. The four momenta $(p_x, p_y, p_z, iE/c)$ in K are

$$p_x = \frac{m_0 v_x}{\sqrt{1-\beta^2}}, \quad p_y = \frac{m_0 v_y}{\sqrt{1-\beta^2}}, \quad p_z = \frac{m_0 v_z}{\sqrt{1-\beta^2}}, \quad E = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \qquad \text{(A - 1)}$$

with $\beta = v/c$ (v is the velocity in K) and $v_z \simeq v$. Similar relations hold for the four-momenta $(p_x', p_y', p_z', iE'/c)$ measured in K′.

In K the longitudinal momentum deviation from the design value is

$$(\Delta p_z) \equiv p_z - p_{z0}$$

$$= m_0 c\left[\frac{\beta}{\sqrt{1-\beta^2}} - \frac{\beta_0}{\sqrt{1-\beta_0^2}}\right]. \qquad \text{(A - 2)}$$

Transformed into K′, the momentum p_z' is written in the form

$$p_z' = \frac{p_z + i\beta_0 \cdot iE/c}{\sqrt{1-\beta_0^2}}$$

$$= \frac{m_0 c}{\sqrt{1-\beta_0^2}} \cdot \frac{\beta-\beta_0}{\sqrt{1-\beta^2}}.$$

Therefore the longitudinal momentum deviation in K′ is written as

$$(\Delta p_z)' = \frac{m_0 c}{\sqrt{1-\beta_0^2}} \cdot \frac{\beta-\beta_0}{\sqrt{1-\beta^2}} \qquad \text{(A - 3)}$$

We define a small parameter ϵ,

$$\epsilon \equiv \beta - \beta_0 \qquad \text{(A - 4)}$$

We expand the right-hand sides of Eq.(A-2) and (A-3) in ϵ and retain only the lowest order terms. We then get the expressions for the longitudinal momentum deviation

$$(\Delta P_z) = \frac{m_0 c \epsilon}{(1-\beta_0^2)^{3/2}} \tag{A - 5}$$

$$(\Delta P_z)' = \frac{m_0 c \epsilon}{(1-\beta_0^2)}, \tag{A - 6}$$

so that

$$(\Delta P_z)' = \frac{(\Delta P_z)}{\gamma}. \tag{A - 7}$$